

General instructions. Answer the problems given in the rest of this document. Edit this LaTeX template to add your answers, as described in the file named `00_README.txt`, then compile it to produce `homework_11.pdf`.

Don't know? Say so. If you don't know the answer to a problem or any part of the problem, we encourage you to simply write “*I don't know*” instead of giving an answer. (If you write “I don't know” you can also add any questions you might have.)

Gentle reminder: homework is for learning. As noted in [Lecture Note 1](#), engaging fully with the homeworks is the best way to learn the material. *The primary role of homeworks is to help you learn the material, and the primary role of feedback on homeworks is to help you improve your ideas.*

Please try to explain your ideas as clearly as you can. The clearer your explanations are, the more useful the feedback on them can be.

Collaboration and citation policy. If you are doing this homework as part of a course, follow the course's collaboration and citation policy. In any case, at the end of each answer, list the people you collaborated with, and cite every source that your answer uses ideas from (other than the lecture notes and the sources listed in the lecture notes). If there are none, write “none”.

Problem 1. (Sections 8.1 and 8.2)

Several families are having a dinner party. To be more social, they would like to arrange seating so that no two members of the same family are at the same table. There are p families and q tables. For each $i \in \{1, 2, \dots, p\}$, family i has a_i members, where $a_i \leq q$. For each $j \in \{1, 2, \dots, q\}$, table j has b_j seats, where $b_j \leq p$. The goal is to determine if there exists a seating arrangement (assigning each family member to a seat) such that no two guests are assigned to the same seat, and no two members of the same family sit at the same table. Here is the formal definition of the problem, as a decision problem:

Family Seating

input: positive integers $p, q, a_1, \dots, a_p$, and $b_1, \dots, b_q$, where $\max_i a_i \leq q$ and $\max_j b_j \leq p$.

output: ‘Yes’ if it is possible to assign each guest to a seat such that (1) no two guests are assigned the same seat and (2) each table has at most one member of each family. If it is not possible, output ‘no’.

Recall the Max Flow decision problem:

input: digraph $G = (V, E)$ with edge capacities, source and sink $s, t \in V$, and a value $k \geq 0$

output: ‘Yes’ if G has an s - t flow of value at least k , otherwise ‘no’.

Develop a poly-time reduction r from Family Seating to Max Flow.

Hint: First study the reduction from Bipartite Matching to Max Flow.

- Define your reduction r by defining, for every possible instance $I = (p, q, a, b)$ of Family Seating, what output $(G, s, t, k) = r(I)$ the reduction r gives. Your definition must state precisely how G , s , t , and k are defined for the given (p, q, a, b) .
- Show that your reduction is a *polynomial-time* reduction, by explaining carefully how, given I , one can compute the output $r(I)$ in time polynomial in the input size $(p + q)$.
- Consider the example input I with $p = 2$, $q = 3$, $a = (2, 2)$, $b = (1, 2, 2)$. Given this specific I , calculate the output $(G, s, t, k) = r(I)$. Draw the resulting flow network. What is k ?
- For your reduction r to be correct, it must be that, *for every Family Seating instance I , the answer to I is ‘yes’ if and only if the answer to the Max Flow decision problem $r(I)$ is ‘yes’*. Show the “if” direction. Given an arbitrary Family Seating instance I , let $G = (V, E)$, s, t , and k be the output of your reduction r given I . Describe carefully how, given any s - t flow f in G of value at least k , one can obtain a valid seating arrangement for I . (Assume that, if all edge capacities in G are integers, then on each edge $(u, w) \in E$ the flow $f(u, w)$ is an integer.)
- For the input I from Part (c), the answer is yes, so the flow network $(G, s, t) = r(I)$ should have an (integer) flow of value at least k . Draw such an flow in your flow network. What is the seating arrangement corresponding to that flow?
- Now show the “only if” direction. Given an arbitrary Family Seating instance I , let $G = (V, E)$, s, t , and k be the output of r given I . Describe carefully how, given any valid seating arrangement for I , one can obtain an s - t flow f in G of value at least k .

Problem 1 answer

(a) Fix an instance $I = (p, q, a = (a_1, a_2, \dots, a_p), b = (b_1, b_2, \dots, b_q))$ of Family Seating. The output of the reduction r on input I is $r(I) = (G = (V, E), s, t, k)$ where digraph $G = (V, E)$ with edge capacities, source s , sink t , and value k are defined as follows.

★...★

(b) ★...★

(c) For the input I with $p = 2, q = 3, a = (2, 2), b = (1, 2, 2)$, the output $r(I)$ of the reduction is as follows. The value of k is ★...★.
Here's a drawing of the flow network G :

★...★

Problem 1 answer, continued

(d) Fix an arbitrary Family Seating instance $I = (p, q, a = (a_1, \dots, a_p), b = (b_1, \dots, b_q))$. Let $G = (V, E)$, s, t , and k be the output $r(I)$ of the reduction r given I . Assume that G has an integer s - t flow of value at least k . Let f be such a flow. From f , a valid seating arrangement X for I can be obtained as follows:

For each family $i \in \{1, \dots, p\}$ and table $j \in \{1, \dots, q\}$, make the number of members of family i sitting at table j be $\star \dots \star$.

(e) For the flow network in Part (c), a flow of value at least k is as follows:

The value of k is $\star \dots \star$.

Here is a drawing of a flow of value k in that network:

$\star \dots \star$

A corresponding seating arrangement is as follows:

$\star \dots \star$

Problem 1 answer, continued

(f) Fix an arbitrary Family Seating instance $I = (p, q, a = (a_1, \dots, a_p), b = (b_1, \dots, b_q))$. Let $G = (V, E)$, s, t , and k be the output $r(I)$ of the reduction r given I . Assume that I has a valid seating arrangement. Let X be such an arrangement. From X , an s - t flow f of value at least k in G can be obtained as follows.

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Problem 1 collaboration and citations

I collaborated on this problem with the following people:

● ★...★

My answers use ideas from the following sources (other than the lecture notes and textbooks):

● ★...★

Problem 2. (Section 8.2; Lecture Note 9)

K -Path is the following problem:

input: A tuple $(G = (V, E), s, t, k)$, where G is a directed graph with non-negative integer edge weights, $s, t \in V$ are two vertices, and k is a non-negative integer.

output: ‘Yes’ if G has an s - t path P with $\text{wt}(P)$ equal to k , otherwise ‘no’.

Recall that Subset Sum is the following problem:

input: Sequence $x = (x_1, x_2, \dots, x_n)$ of non-negative integers and a non-negative integer T .

output: ‘Yes’ if there a subset $S \subseteq \{1, \dots, n\}$ such that $\sum_{i \in S} x_i = T$, otherwise ‘no’.

Describe a polynomial-time reduction from Subset Sum to k -Path.

- Define your reduction r by defining, for every possible instance $(x = (x_1, x_2, \dots, x_n), T)$ of Subset Sum, what output $r(x, T)$ the reduction gives. The output will be a tuple $(G = (V, E), s, t, k)$. Your definition must state precisely how G , s , t , and k are defined for the given (x, T) . *Hint: Construct G so its s - t paths correspond to the subsets $S \subseteq \{1, \dots, n\}$.*
- Show that your reduction is a *polynomial-time* reduction, by explaining carefully how, given (x, T) , one can compute the output $r(x, T) = (G, s, t, k)$ in time polynomial in the input size $\Theta(n \log T)$.
- Consider the example input (x, T) with $x = (5, 9, 10, 20)$ and $T = 29$. Calculate the output $(G, s, t, k) = r(x, T)$ for this input. What is k ? Draw the resulting edge-weighted digraph.
- For your reduction r to be correct, it must be that, *for every Subset Sum instance (x, T) , the answer to (x, T) is ‘yes’ if and only if the answer to the k -Path instance $r(x, T)$ is ‘yes’.*
Show the “if” direction. Given an arbitrary Subset Sum instance (x, T) , let G , s , t , and k be the output of your reduction r given (x, T) . That is, $(G, s, t, k) = r(x, T)$. Describe carefully how, given any s - t path P in G with $\text{wt}(P) = k$, one could obtain a subset $S \subseteq \{1, \dots, n\}$ such that $\sum_{i \in S} x_i = T$.
- For the input (x, T) from Part (c), the answer is ‘yes’, so, for $(G, s, t, k) = r(x, T)$, there should be an s - t path in G of weight k . What is such a path P in G ? What is the subset $S \subseteq \{1, 2, 3, 4\}$ corresponding to this path P ?
- Now show the “only if” direction. Given an arbitrary Subset Sum instance (x, T) , let $G = (V, E)$, s , t , and k be the output of your reduction r given (x, T) . That is, $(G, s, t, k) = r(x, T)$. Describe carefully how, given any subset $S \subseteq \{1, \dots, n\}$ such that $\sum_{i \in S} x_i = T$, one could obtain from S an s - t path P in G with $\text{wt}(P) = k$.

Problem 2 answer

(a) Fix an arbitrary Subset Sum instance $(x = (x_1, x_2, \dots, x_n), T)$. The output of the reduction r on input (x, T) is $r(x, T) = (G, s, t, k)$ where G , s , t and k are defined as follows.

★...★

(b) ★...★

(c) For the input (x, T) with $x = (5, 9, 10, 20)$ and $T = 29$, the output $r(x, T)$ of the reduction is as follows. The value of k is ★...★.

Here is a drawing of the graph G :

★...★

Problem 2 answer, continued

(d) Fix an arbitrary Subset Sum instance $(x = (x_1, x_2, \dots, x_n), T)$. Let $G = (V, E)$, s, t , and k be the output $r(x, T)$ of the reduction r given (x, T) . Assume that G has an s - t path of weight k . Let P be such a path. From P , a subset $S \subseteq \{1, \dots, n\}$ such that $\sum_{i \in S} x_i = T$ can be obtained as follows.

★...★

(e) The value of k is ★...★.

An s - t path of weight k is $P = (\text{★...★})$

The corresponding subset S is $S = \{\text{★...★}\}$

(f) Fix an arbitrary Subset Sum instance $(x = (x_1, \dots, x_n), T)$. Let $G = (V, E)$, s, t , and k be the output $r(x, T)$ of the reduction r given (x, T) . Assume that there is a subset $S \subseteq \{1, \dots, n\}$ such that $\sum_{i \in S} x_i = T$. From S , an s - t path P with $\text{wt}(P) = k$ can be obtained as follows.

★...★

Problem 2 collaboration and citations

I collaborated on this problem with the following people:

● ★...★

My answers use ideas from the following sources (other than the lecture notes and textbooks):

● ★...★